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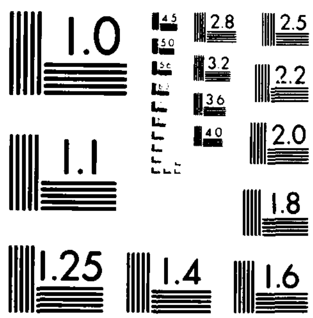

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THESIS

OPTIMAL SOLUTIONS UNDER RISK AND
UNCERTAINTY FOR SEVERAL VARIATIONS
OF THE NEWSBOY PROBLEM

by

Sutat Khayim

March 1980

Thesis Advisor: G.F. Lindsay

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Optimal Solutions Under Risk and Uncertainty For Several Variations Of The Newsboy Problem

by

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Submitted in partial fulfillment of the requirements for the degree of

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ABSTRACT

Four variations of the well-known newsboy problems are investigated for their optimal solutions using several principles of choice. Two "mixed" newsboy problems which are addressed are the continuous supply, discrete demand problem, and the discrete supply, continuous demand problem. Additionally, two "package supply" newsboy problems are investigated. For all four problems, optimal solutions are given to minimize expected cost, maximize the probability that cost is below an aspiration level, meet the LaPlace criterion, and minimax cost.

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I. INTRODUCTION

The newsboy problem has been known for many years. It is named for the problem faced daily by a boy selling newspaper on a street. If he buys too many papers for resale he will have papers left. If, on the other hand, he buys too few papers he will not have enough papers for his customers. Both cases cost him losses. Thus his decision problem each day is to select the number of papers to buy in order to minimize his losses. The concept of the newsboy problem has been developed and applied to many real world problems. In operations research texts, this problems is called A Stochastic Single-Period Inventory Model. The problem is often modified in order to be appropriate to the specific situation of interest.

This thesis is devoted to four variations on the newsboy problem. In the standard newsboy problem, supply and demand have the same units, and either both are discrete or both are continuous. We shall be concerned with cases where there are differences in the characteristics of the demand and supply variables. For examples, the case where demand is discrete but supply is continuous, and the opposite case, are "mixed newsboy problems" whose optimal solutions are investigated in this paper.

The assumptions[1] for these models are still the same as those of the classic single-period model. The quantities

left at the end of the last period are not allowed to carry over to the next period. Orders are placed once for each period. When demand exceeds supply a shortage cost is incurred, and when supply exceeds demand a surplus cost is incurred.

The four variations of the newsboy problem we shall examine in this thesis are the cases where

(1) demand is continuous when supply is discrete,

(2) demand is discrete when supply is continuous,

(3) demand and supply are both discrete but supply is in lots of n items, and

(4) demand and supply are both continuous but supply is in lots of size n .

These four cases may serve as models for a number of real world problems. Some examples for these mixed newsboy problems are: the stock of one gallon cans of lubricating oil for customer whose needs depend upon the size of engine, the butcher who stocks live cattle to be sold in pieces, coffee prepared in a pot but sold by the cup, stock of sugar to be sold in one-pound bags, and eggs stocked in trays but sold individually.

The purpose of this thesis is to find optimal decision rules for these models which are appropriate for application. We will begin in Chapter II by reviewing the notion of decision making, in terms of decisions under risk and under uncertainty. For risk we will consider two principles of choice, namely expected cost[3] and aspiration level[2]. For

uncertainty we will consider two principles of choice, namely LaPlace[2] and minimax cost[2]. In the second part of Chapter II we will display the optimal decision rules for the standard newsboy problem that result from applying these four principles of choice. Before closing Chapter II, we will introduce and state the four mixed newsboy problems which are the subject of this thesis.

In Chapter III through Chapter VI we will investigate each of the four variations of the newsboy problem and derive formulas of optimal decision rules employing the expected cost, aspiration level, LaPlace, and minimax cost principles of choice. In some cases we will propose alternate approaches for optimal solutions. Conclusions and suggestions for further work will be given in Chapter VII.

II. STATEMENT OF THE PROBLEM

In solving newsboy problems we become involved in decision-making. The decision-maker seeks a value for supply Q that is most appropriate for each situation, and the consequences of his decision depend upon the future occurrence of demand which may be certain, estimated, or unknown. We seek optimal decision rules for various information environments.

This chapter has three major parts. First, in order to understand the four following chapters more easily, we review the notions of decisions under risk and under uncertainty, give their short definitions and explain some principles of choice as ways to approach optimal decision rules. Secondly, we present the two classic newsboy problems where both demand and supply are discrete and where both demand and supply are continuous, displaying the optimal decision rules for both problems under risk and under uncertainty. Finally, we introduce and describe the four related problems which are the subject of this thesis.

A. DECISIONS UNDER RISK AND UNDER UNCERTAINTY

Good solutions to a decision problem depend upon the amount of information we have about the future; in the newsboy problem, we are concerned about the magnitude of demand D. Clearly, if the magnitude of demand were known in advance with certainty, the decision problem would be

easy, i.e., the "optimal" choice of a value for the decision variable Q would be equal to demand D so that no costs relating to the decision would be incurred. For real-world problems, we usually do not know the demand value exactly, and have instead varying amounts of information about what it might be, perhaps expressed as the probabilities of occurrence of various values of demand D. Decisions have been grouped according to the information we have or can estimate about the future. They are classified as decisions under certainty, under risk and under uncertainty[2].

1. Decisions Under Risk

A decision under risk is one in which we are able to estimate the probabilities of future states occurring. Thus we see that the presence of probabilities is the key to the definition of decisions under risk. The manner in which this probabilistic information is used may vary in many different ways which are called principles of choice. In this thesis we will consider two principles which are frequently used, expectation and aspiration level.

Expectation. The expectation principle suggests that we select the alternative whose expected payoff is the most favorable. In other words, we choose the alternative so as to maximize the expected profit or minimize the expected cost. In this paper minimizing expected cost is more interesting and will have the principle roles in every chapter.

Aspiration Level. The use of the expectation or the average is most appropriate in cases where the decision or

choice is to be made many times. If decisions will be made only one time or a few times the use of expected values has very little meaning, and the aspiration level principle is more appropriate and practical[2]. Suppose a decision maker has some aspiration level A about return. He would be satisfied with a profit of at least A or in a cost interpretation, a cost of at most A. The aspiration level principle suggests that we choose the alternative which maximizes the probability $P\{\text{Profit} \geq A\}$ if the return is the profit, or which maximizes the probability $P\{\text{Cost} \leq A\}$ if the return is the cost. Again, in this paper we are interested in the latter case.

2. Decision Under Uncertainty

In many decision problems, the probability distribution of demand may not be achievable. This leads to a decision under uncertainty. A decision under uncertainty is one in which it is felt that the probabilities of future states occurring cannot be estimated. The situation which is new or the familiar situation in a new environment are typical situations of decisions under uncertainty. For example, uncertainty about demand occurs if the number of spares to be carried must be determined without information about the reliability of the original and spares.

If the probability distribution of demand is unknown, we might still find other information about the probability function such as mean, mode, median, skewness and range. In a case where the range of demand is known, three principle

of choice which have been proposed are the LaPlace, minimax cost (or maximin profit) and minimax regret principles. We will be concerned only with the LaPlace and minimax cost principles in this thesis.

LaPlace Principle. Given a value for the maximum demand that might occur, this principle suggests that the probabilities of demand may be assumed as uniform. We then choose the alternative in which the expected cost is minimized.

Minimax Cost Principle. This principle is called a pessimistic[2] approach. It suggests that we choose so as to minimize the maximum cost (minimax) or, in other words, we choose so as to make the worst outcome as desirable as possible.

The next section will introduce the standard newsboy problems, showing rules for obtaining the optimal solutions.

B. THE CLASSIC NEWSBOY PROBLEM

The newsboy problem appears frequently in a variety of scenarios, but in this section we will propose only the classic problems in order to compare to the "mixed" newsboy problems and the "package supply" newsboy problems which will be discussed in subsequent chapters. We will give the characteristics of the problem first, construct the cost function, and display the well-known decision rules to find the optimal solutions for decisions under risk and under uncertainty. We will begin with the continuous case and follow with the discrete case.

The classic newsboy problems are for the cases where demand D and supply Q are both continuous or both discrete. They deal with a single commodity, and with decision over a single time period. The quantities left at the end of the period are not allowed to carry over the next period. The decision maker has an opportunity to order only at the beginning of each period. Either demand in excess of supply or supply in excess of demand cause the costs to the supplier. Some real world problems are the stocking of spare parts, perishable items, style goods and special seasonal items[1].

The Newsboy Cost Equation. We construct the newsboy cost equation as follows. Let

Q = Supply; the number of items (or the amount) the decision-maker orders for each period. This is the decision variable for the problem,

D = Demand; the number of items (or the amount) which will be needed by customers in one period,

Q^* = Optimal supply quantity,

C_s = Unit cost of surplus; the cost for each unit by which supply Q exceeds demand D,

C_0 = Unit cost of shortage; the cost for each unit by which demand D exceeds supply Q,

$f(D)$ = Probability density function of demand D.

$F(D)$ = Distribution function for $f(D)$,

$C(Q)$ = Total cost, and

$E\,C(Q)$ = The expected total cost.

With this notation we can write the cost function as

$$C(Q) = \begin{cases} C_s(Q - D) , & 0 \leq D \leq Q , \\ \\ C_0(D - Q) , & Q < D . \end{cases} \tag{1}$$

Optimal decision rules for decisions under risk and under uncertainty are presented in the next two sections. We will consider first the newsboy problem where supply and demand are both continuous, and then the case where both supply and demand are discrete.

1. <u>The Continuous Case</u>

When supply and demand are continuous variables the expected cost is

$$E\{C(Q)\} = \int_0^{\infty} (\text{cost}) f(D) dD ,$$

or

$$E\{C(Q)\} = \int_0^{Q} C_s(Q - D) f(D) dD + \int_Q^{\infty} C_0(D - Q) f(D) dD . \tag{2}$$

We can find Q^* that minimizes $E\{C(Q)\}$ by differentiating $E\{C(Q)\}$; we obtain the well-known result

$$F(Q^*) = \frac{C_0}{C_s + C_0}, \qquad (3)$$

as the optimal decision rule.

For an aspiration level solution, we let A be the cost aspiration level or a maximum acceptable cost value. We seek the value of supply Q such that the probability that cost would be less than or equal to A is maximized, or a value of Q providing $\max_{Q}\{P(\text{Cost} \le A)\}$. With some effort, we can obtain

$$P\{\text{Cost} \le A\} = F\{Q + A/C_0\} - F\{Q - A/C_s\}.$$

Assuming F(D) is everywhere differentiable, we set

$$\frac{dP(\text{Cost} \le A)}{dQ} = 0,$$

yielding

$$f(Q^* + A/C_0) = f(Q^* - A/C_s), \qquad (4)$$

as the decision rule for optimal solution.

When we do not have an estimate of the probability distribution of demand the newsboy problem may be treated as a decision under uncertainty. We assume that an estimate of maximum demand D_{max} is available. Then, for the LaPlace principle of choice, we assume

$$f(D) = \frac{1}{D_{max}}, \quad 0 \le D \le D_{max},$$

and it follows that

$$F(D) = \frac{D}{D_{max}} ,$$

From the expected cost rule (3) we can obtain

$$Q^* = \left(\frac{C_0}{C_s + C_0}\right) D_{max} \tag{5}$$

as the optimal decision rule.

We can minimize the maximum cost when supply Q is chosen so that the cost at demand D = 0 is equal to the cost at demand $D = D_{max}$, or

$$C_s Q^* = C_0 (D_{max} - Q^*) ,$$

which yields

$$Q^* = \left(\frac{C_0}{C_s + C_0}\right) D_{max} \tag{6}$$

as the decision rule. This minimax cost result is identical to that given by the LaPlace rule (5).

2. The Discrete Case

When demand is a discrete random variable and supply is limited to value of demand, the cost function of this classic discrete case is

$$C(Q) = \begin{cases} C_s(Q - D), & D = 0, 1, 2, \ldots, Q \\ \\ C_0(D - Q), & D = Q+1, Q+2, \ldots, \end{cases} \quad (7)$$

and the expected cost is

$$E\{C(Q)\} = \sum_{D=0}^{Q} C_s(Q - D)f(D) + \sum_{D=Q+1}^{\infty} C_0(D - Q)f(D). \quad (8)$$

Since supply Q and demand D are discrete, we apply the notion of sufficient conditions for a local minimum,

$$E\{C(Q^*)\} < E\{C(Q^* + 1)\}$$

and

$$E\{C(Q^*)\} < E\{C(Q^* - 1)\},$$

to the expected cost equation (8). Simplifying, we finally obtain

$$F(Q^* - 1) < \frac{C_0}{C_s + C_0} < F(Q^*) \quad (9)$$

as the optimal decision rule to minimize expected cost.

For the Aspiration Level principle of choice, we have to choose supply Q so as to keep the cost below a desired value A with the maximum probability, which can be interpreted as

$$\underset{Q}{\text{Max}}\{P(\text{Cost} \le A)\} .$$

We can obtain Q* by constructing the cost matrix associated with various combinations of supply Q and demand D as shown in Table I below. From this table we compute the probability that cost will not exceed A for each value of Q and select as Q* the value of Q that provides a maximum probability.

TABLE I. The Cost Array for the Classic Discrete Case.

	f(D)	f(0)	f(1)	f(2)	D f(3)				
	D	0	1	2	3	.	.	.	.
	0	0	C_0	$2C_0$	$3C_0$	.	.	.	.
	1	C_s	0	C_0	$2C_0$	.	.	.	.
	2	$2C_s$	C_s	0	C_0	.	.	.	.
Q	3	$3C_s$	$2C_s$	C_s	0	.	.	.	.
	.	.	.	.	.	.	.	.	.
	.	.	.	.	.	.	.	.	.

When there is uncertainty about demand, the LaPlace approach would assume $f(D) = \frac{1}{D_{max} + 1}$, $D = 0, 1, 2, \ldots D_{max}$, and it follows that

$$F(D) = \frac{D + 1}{D_{max} + 1} .$$

Substituting F(D) in the optimal decision rule to minimize Expected Cost (9), we obtain

$$Q^* < \left(\frac{C_0}{C_s + C_0}\right)(D_{max} + 1) < Q^* + 1 \qquad (10)$$

as the optimal decision rule.

In order to minimize maximum costs at Q*, we use the notions of

$$\text{Maximum } C(Q^*) < \text{Maximum } C(Q^* + 1)$$

and

$$\text{Maximum } C(Q^*) < \text{Maximum } C(Q^* - 1) .$$

We know that the maximum value of the cost function (6) is either C_sQ^* or $C_0(D_{max} - Q^*)$. Applying this information to the two conditions above, we have

$$Q^* < \frac{C_0}{C_s + C_0}(D_{max} + 1) < Q^* + 1 \qquad (11)$$

as the optimal decision rule, which is identical to the one of LaPlace solution found earlier (10).

C. FOUR RELATED PROBLEMS

This thesis proposes four newsboy-like problems to be examined. All of them deal with daily real life problems

and they have small variations from the classic newsboy problems. We shall now investigate the newsboy problems where

(1) demand D is continuous and supply Q is discrete;

(2) demand D is discrete and supply Q is continuous;

(3) demand D and supply Q are both discrete but supply Q is in lots of n items, where n is a fixed integer; and

(4) demand D and supply Q are both continuous but supply Q is in lots of size n, where n is a fixed real number.

The first two problems could be called "mixed newsboy problems" and the last two "package supply" problems. We will describe these problems one by one in the following sections.

1. <u>Demand Is Continuous, Supply is Discrete</u>

The case of continuous demand and discrete supply might be appropriate for any number of real world problems where demand D can be any real number from zero to infinity, and supply Q is acquired in containers or blocks such that the supply quantity is an integer. The units of demand D and supply Q are the same. This case will be investigated in detail in Chapter III.

2. <u>Demand Is Discrete, Supply Is Continuous</u>

A situation opposite to the preceding one is the case of discrete demand and continuous supply. Here the amount of demand D is in, say, weight units or volume units which are

integer-valued. The amount of supply Q is any real number from zero to infinity which has the same units as demand D. We will examine this case later in Chapter IV.

3. Demand And Supply Are Both Discrete But Supply Is In Lots of n Items

This case is as the same as the classic discrete one, except that supply Q is in Lots of n items, where n is fixed integer number. Here supply must be acquired in lots of n items, as in a can, box or tray, and demand D is in individual items. We will call this case a "package supply" newsboy problem. If we let S represent the number of packages ordered, S = 0, 1, 2,..., then the supply Q is equal to nS, where n is a package size. We will discuss and investigate optimal decision rules for this problem in Chapter V.

4. Demand And Supply Are Both Continuous, But Supply Is In Lots of Size n

This case is similar to the first case where demand D is continuous and supply Q is discrete. The difference is that supply Q in this case is served in a container whose capacity is not in one unit. In other words, the package size n can be any non-negative real number. We can see that if we let n be unity, this case is identical to the first case. Note that the supply Q is equal to nS, where S represents the number of packages ordered. We will discuss this package supply problem in detail later in Chapter VI.

In the subsequent four chapters, we seek optimal decision rules both under risk and under uncertainty for the four cases described above.

III. OPTIMAL SOLUTIONS WHERE DEMAND IS CONTINUOUS AND SUPPLY IS DISCRETE.

In Chapter II we introduced the classic newsboy problems; giving optimal solutions for expected cost, aspiration level, LaPlace and minimax cost. In the classic problem, demand and supply have the same units. This chapter we will examine the special case where demand is continuous but supply is discrete. The optimal decision rules for this case may differ from the classic one because of its nature. For this mixed newsboy problem, we will investigate the nature of optimal solutions when the distribution of demand can be estimated (risk), and then derive formulas to determine optimal supply quantities for case where the demand distribution cannot be estimated (uncertainty).

We will consider two alternate ways in which a known or estimated probability distribution of demand could be used: (i), to determine a value of variable Q in which the expected (average) cost is minimized, and (ii), to maximize the probability of keeping cost below some aspiration level A. Before looking at the expected cost case we construct the cost function as follows. Let

$$Q = \text{Supply}, \quad Q = 0, 1, 2, \ldots ,$$

and

$$D = \text{Demand}, \quad 0 \leq D < \infty .$$

The cost function is

$$C(Q) = \begin{cases} C_s(Q - D) , & 0 \le D \le Q , \quad Q = 0, 1, 2, \ldots \\ C_0(D - Q) , & Q < D , \end{cases} \qquad (12)$$

where C_s is the cost per unit of supply that exceeds demand and C_0 is cost per unit of demand that exceeds supply. The cost function is shown in Figure 1.

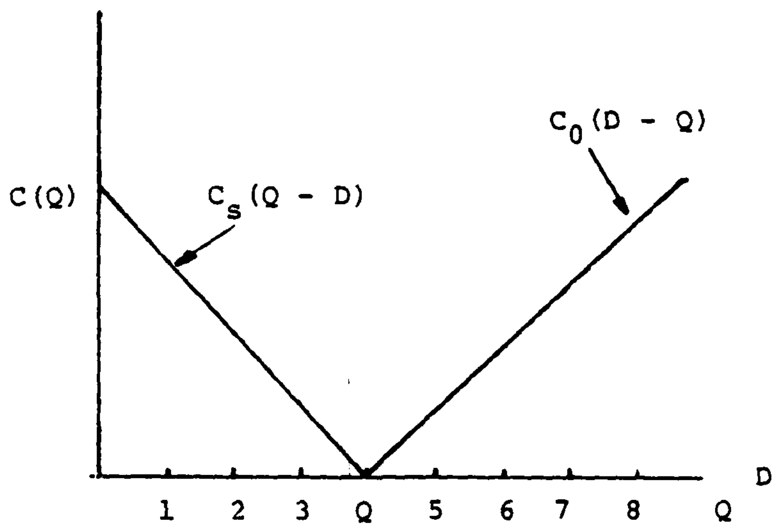

FIGURE 1. The Cost Function Of Newsboy Problem where Demand is Continuous and Supply is Discrete.

A. MINIMIZING EXPECTED COST

From the cost function (12) we can express the expected cost equation as follows. Since demand D is continuous with probability density f(D),

$$E\{C(Q)\} = \int_0^Q C_s(Q - D)f(D)dD + \int_Q^\infty C_0(D - Q)f(D)dD . \qquad (13)$$

To find optimal solution Q^* for the least expected cost we cannot differentiate $E\{C(Q)\}$ with respect to Q since Q is discrete. We will use two sufficiency conditions for a local minimum of a function. If

$$E\{C(Q^*)\} < E\{C(Q^* + 1)\} ,$$

and

$$E\{C(Q^*)\} < E\{C(Q^* - 1)\} ,$$

then the expected cost function has a local minimum at Q^*. Expanding the first inequality with the expected cost function (13) yields:

$$\int_0^{Q^*} C_s(Q^*-D)f(D)dD + \int_{Q^*}^{\infty} C_0(D-Q^*)f(D)dD < \int_0^{Q^*+1} C_s(Q^*+1-D)f(D)dD + \int_{Q^*+1}^{\infty} C0(D-Q^*-1)f(D)dD . \qquad (14)$$

The right hand side of this inequality (14) may be expanded as

$$\int_0^{Q^*+1} C_s(Q^*-D)f(D)dD + \int_0^{Q^*+1} C_s f(D)dD + \int_{Q^*+1}^{\infty} C_0(D-Q^*)f(D)dD - \int_{Q^*+1}^{\infty} C_0 f(D)dD ,$$

and therefore we can simplify the inequality (14) to

$$\int_{Q^*}^{Q^*+1} C_0 (D-Q^*) f(D)\,dD + \int_{Q^*+1}^{\infty} C_0 f(D)\,dD < \int_{Q^*}^{Q^*+1} C_s (Q^*-D) f(D)\,dD + \int_{0}^{Q^*+1} C_s f(D)\,dD \ ,$$

which is

$$\int_{Q^*}^{Q^*+1} C_0 (D-Q^*) f(D)\,dD + C_0\{1-F(Q^*+1)\} < -\int_{Q}^{Q^*+1} C_s (D-Q^*) f(D)\,dD + C_s F(Q^*+1) \ .$$

Finally, after simplifying we have

$$F(Q^*+1) - \int_{Q^*}^{Q^*+1} (D-Q^*) f(D)\,dD > \frac{C_0}{C_s + C_0} \qquad (15)$$

as a part of the decision rule.

In the same way, expanding the second sufficiency condition with the expected cost function (13) yields:

$$\int_{0}^{Q^*} C_s (Q^*-D) f(D)\,dD + \int_{Q^*}^{\infty} C_0 (D-Q^*) f(D)\,dD$$

$$< \int_{0}^{Q^*-1} C_s (Q^*-1-D) f(D)\,dD + \int_{Q^*-1}^{\infty} C_0 (D-Q^*+1) f(D)\,dD. \ (16)$$

We expand and simplify this inequality (16) in the same manner as we did for the first condition, and finally, we have

$$F(Q^*-1) + \int_{Q^*+1}^{Q^*} (Q^*-D)\, f(D)\, dD < \frac{C_0}{C_s + C_0} \, . \qquad (17)$$

The inequalities (15) and (17) lead to

$$F(Q^*-1) + \int_{Q^*-1}^{Q^*} (Q^*-D)\, f(D)\, dD < \frac{C_0}{C_s + C_0} < F(Q^*+1) - \int_{Q^*}^{Q^*+1} (D-Q^*)\, f(D)\, dD \qquad (18)$$

as the optimal decision rule for minimizing expected cost for this mixed newsboy problem. For determining optimal solution Q^*, this decision rule may not be tractable because of difficulties in evaluating the integrals.

As alternative approach, we can determine the optimal solution for the minimizing expected cost as follows:

(1) Treat demand D and supply Q as the classic continuous case and find the optimal solution Q^* from decision rule (3)

$$F(Q^*) = \frac{C_0}{C_s + C_0} \, .$$

If Q^* is an integer, it is the optimal solution for this mixed newsboy problem. If this Q^* is not an integer, we proceed as follows.

(2) Let Q_1^* be the largest integer which is less than Q^*.

(3) Let $Q_2^* = Q_1^* + 1$.

As this point we know that the optimal solution is either Q_1^* or Q_2^* .

(4) Find the expected cost of Q_1^* and Q_2^* from (2):

$$E\{C(Q)\} = \int_0^Q C_s(Q-D)f(D)\,dD + \int_Q^\infty C_0(D-Q)f(D)\,dD.$$

The optimal ordered quantity is the value which provides the smaller expected cost. Here again, an optimal solution requires that the integrals be evaluated.

B. ASPIRATION LEVEL SOLUTION

We seek the optimal decision rule with the aspiration level principal of choice for this mixed newsboy problem where demand D is continuous and supply Q is discrete. Following the same notion as in the classic continuous case (as shown in Figure 2), we see that the optimal Q^* is such that $P\{\text{Cost} \leq A\} = F\{Q^* + A/C_0\} - F\{Q^* - A/C_s\}$ is maximized. We cannot differentiate $P\{\text{Cost} \leq A\}$ with respect to Q because Q is discrete.

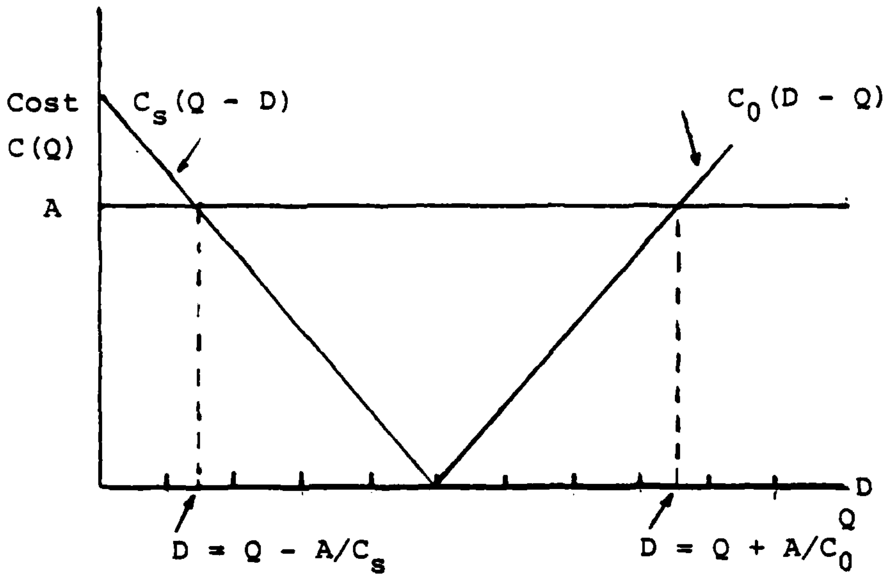

FIGURE 2. Conditions for an Aspiration Level Solution in a Mixed Newsboy Problem when Demand D is Continuous and Supply Q is Discrete.

We know that

$$F(Q + A/C_0) - F(Q - A/C_s) = \int_{Q-A/C_s}^{Q+A/C_0} f(D)\,dD ,$$

and we can construct a table to find Q* such that $\int_{Q^*-A/C_s}^{Q^*+A/C_0} f(D)\,dD$ is a maximum. This is shown in Table II.

TABLE II. Illustration of the Method to find an Optimal Aspiration Level Solution Q* for the Mixed Newsboy Problem Where Demand D is Continuous and Supply Q is Discrete, Using Hypothetical Data.

	Q	$\int_{Q-A/C_0}^{Q+A/C_0} f(D)\,dD$	
	0	.1	
	1	.3	
	2	.6	
Q* = 3	→ 3	.8 ←	maximum
	4	.4	

For this particular mixed newsboy problem, we have discussed two principles of choice and presented decision procedures for each principle. Those procedures can be applied only the case where the probability distribution of demand is known or can be estimated. We now seek a decision rule for the uncertainty case where we cannot estimate the demand probabilities. The two following principles of choice which will be discussed in connection with uncertainty are the LaPlace and minimax cost principles.

C. LAPLACE SOLUTION

As described in Chapter II, the LaPlace principle assumes that the probability density function of demand is

uniformly distributed over the range from zero to D_{max}. Thus for continuous D we have

$$f(D) = \frac{1}{D_{max}}, \quad 0 \leq D \leq D_{max},$$

and the distribution function is

$$f(D) = \frac{1}{D_{max}}.$$

The optimal expected cost rule (18) developed at the beginning of this chapter is

$$F(Q^*-1) + \int_{Q^*-1}^{Q^*} (Q^*-D)f(D)\,dD < \frac{C_0}{C_s + C_0} < F(Q^*+1) - \int_{Q^*}^{Q^*+1} (D-Q^*)f(D)\,dD.$$

Substituting F(D) and f(D) in the above inequality, we have

$$\frac{(Q^*-1)}{D_{max}} + \int_{Q^*-1}^{Q^*} (Q^*-D)\frac{1}{D_{max}}\,dD < \frac{C_0}{C_s + C_0} < \frac{Q^*+1}{D_{max}} - \int_{Q^*}^{Q^*+1} (D-Q^*)\frac{1}{D_{max}}\,dD .$$

This inequality leads to

$$(Q^*-1) + \int_{Q^*-1}^{Q^*} (Q^*-D)\,dD < \left(\frac{C_0}{C_s + C_0}\right) d_{max} < (Q^*+1) - \int_{Q^*}^{Q^*+1} (D-Q^*)\,dD .$$

After resolving the integrals we have

$$(Q^*-1) + \frac{1}{2} < \left(\frac{C_0}{C_s + C_0}\right) D_{max} < (Q^*+1) - \frac{1}{2}$$

which leads to

$$Q^* - \frac{1}{2} < \frac{C_0}{C_s + C_0} (D_{max}) < Q^* + \frac{1}{2} \qquad (19)$$

as the optimal decision rule for LaPlace solution of this mixed newsboy problem.

D. MINIMAX COST SOLUTION.

For a minimax cost solution as described in Chapter II, we wish to minimize the maximum cost. We assume that the only thing we know about the demand is the maximum D_{max}. The cost function for this mixed problem is as in (12),

$$C(Q) = \begin{cases} C_s(Q - D) , & 0 \leq D \leq Q , \quad Q = 0, 1, 2, \ldots \\ C_0(D - Q) , & Q < D . \end{cases}$$

We know that the maximum cost will occur at the extreme demand D, i.e., at demand D equal to zero or to D_{max}. Since supply Q is integer 0, 1, 2,..., we are sometimes not able to choose Q^* such that $C_sQ^* = C_0(D_{max} - Q^*)$ as in the approach of the standard newsboy problem with supply and demand both continuous. Then the following procedures would be applicable.

(1) Treat supply Q as continuous (as is the demand D) and compute Q^* from

$$Q^* = \left(\frac{C_0}{C_s + C_0}\right)(D_{max})$$

as discussed in Chapter II. This is the minimax cost solution to the standard newsboy problem. If Q^* is an integer then Q^* is the optimal order quantity. If Q^* is not an integer we proceed as follows.

(2) Let Q_1^* be the largest integer which less than Q^* and let $Q_2^* = Q_1^* + 1$. The optimal order supply will be either Q_1^* or Q_2^*. For this principle of choice we have to select Q such that $\underset{Q}{\text{Min}} \max\{C_s Q, C_0(D_{max} - Q)\}$. To solve this problem we may construct a table such as TABLE III. below. The procedure may be described through a simple numerical example.

Example 1. Suppose $D_{max} = 10.8$, $C_0 = 2$ and $C_s = 4$. From the standard continuous case solution, $Q^* = \left(\frac{C_0}{C_s + C_0}\right) D_{max} = 3.6$ and the possible solutions to our mixed newsboy problem are $Q_1^* = 3$ and $Q_2^* = 4$. The optimal solution may be identified by checking costs at $D = 0$ and $D = D_{max}$, as illustrated in TABLE III.

TABLE III. Illustration of a Minimax Cost Solution for a Continuous Demand, Discrete Supply Newsboy Problem, Given $C_0 = 2$, $C_2 = 4$ and $D_{max} = 10.8$.

	C_sQ	$C_0(D_{max} - Q)$	max	
$Q_1^* = 3$	12	15.6	15.6	← min.
$Q_2^* = 4$	16	13.6	16	

From TABLE III. above, clearly the optimal solution is $Q_1^* = 3$.

We might consider the another approach for this case as follows. Since Q is integer 0, 1,..., in order for the maximum costs to be minimized at Q^*, the following inequalities must hold:

$$\text{maximum } C(Q^*) < \text{maximum } C(Q^* + 1) \qquad (20)$$

and

$$\text{maximum } C(Q^*) < \text{maximum } C(Q^* - 1) \, . \qquad (21)$$

As before, the cost function will have its maximum value at either C_sQ or $C_0(D_{max} - Q)$, and the first condition (20) may be written as

$$\max\{C_sQ^*, C_0(D_{max} - Q^*)\} < \max\{ C_s(Q^*+1), C_0(D_{max} - Q^*-1)\},$$

while the second condition (21) may be written as

$$\max\{C_s Q^*, C_0(D_{max} - Q^*)\} < \max\{C_s(Q^*-1), C_0(D_{max} - Q^*+1)\} ,$$

These two inequalities are exactly the same as those found in Chapter II in the investigation of classic discrete case, yielding

$$Q^* < \frac{C_0(D_{max} + 1)}{C_s + C_0} < Q^* + 1$$

as the decision rule of the mixed newsboy problem where demand is continuous and supply is discrete. When we apply this decision rule to Example 1 above, we can compute $\frac{C_0(D_{max} + 1)}{C_s + C_0}$ which is equal to 3.93, and thus we can conclude $Q^* = 3$ as before. It is interesting to note that unlike the standard newsboy problem, the LaPlace and minimax cost solution for this mixed problem are not identical.

The mixed newsboy problem investigated in this chapter is only slightly changed from the original problem. In the next chapter we will examine a mixed model where supply is continuous and demand is discrete, which is opposite to the case discussed in this chapter.

IV. OPTIMAL SOLUTIONS WHERE DEMAND IS DISCRETE AND SUPPLY IS CONTINUOUS.

One interesting mixed newsboy problem is the case where demand D is discrete and supply Q is continuous. This problem differs from the classic newsboy problem and is opposite to the one discussed and investigated in the previous chapter. Problems where demand D is discrete and supply Q is continuous are encountered frequently in application. Some examples are; how much gas will be placed on a ship for portable pumps if the demand for gas is in gallon cans, and how much coffee will be prepared in the pot for each day if the demand is in cups. In this chapter, we will investigate optimal decision rules when the probability distribution of demand is estimated (risk), then later derive formulas to determine the optimal supply quantities for cases where the demand distribution can not be estimated (uncertainty).

The cost function of the problem is needed for determining the optimal decision rules in the following sections. We can construct the cost function of this problem as follows. Let

$$Q = \text{Supply}, \quad 0 \leq Q$$

and

$$D = \text{Demand}, \quad D = 0, 1, 2, \ldots .$$

The cost function is

$$C(Q) = \begin{cases} C_s(Q - D), & Q \geq D, \\ C_0(D - Q), & Q < D, \quad D = 0, 1, \ldots, \end{cases} \tag{22}$$

where C_s is cost per unit of supply that exceeds demand and C_0 is cost per unit of demand that exceeds supply. If we assume that demand may be satisfied only in discrete units then we must account for surpluses of less than one unit which could occur even when demand exceeds supply. Let

Q_c = Continuous supply, $0 \leq Q_c$

Q_i = The integer part of Q_c, and

Q_R = $Q_c - Q_i$, i.e., the remainder

Now we have $Q_c = Q_i + Q_R$. TABLE IV. below, shows some examples from the cost function for this particular problem.

TABLE IV. The Cost Array where Supply is Continuous, and Demand is Discrete.

		DISCRETE DEMAND D						
		0	1	2	3	4	5	. . .
Continuous supply Q_c	0	0	C_0	$2C_0$	$3C_0$	$4C_0$	$5C_0$	. . .
	0.5	$0.5C_s$	$C_0 + 0.5C_s$	$2C_0 + 0.5C_s$	$3C_0 + 0.5C_s$	$4C_0 + 0.5C_s$	$5C_0 + 0.5C_s$	. . .
	1.0	C_s	0	C_0	$2C_0$	$3C_0$	$4C_0$	
	1.5	$1.5C_s$	$0.5C_s$	$C_0 + 0.5C_s$	$2C_0 + 0.5C_s$	$3C_0 + 0.5C_s$	$4C_0 + 0.5C_s$	. . .
	2.0	$2.C_s$	C_s	0	C_0	$2C_0$	$3C_0$	. . .
	2.5	$25C_s$	$1.5C_s$	$.5C_s$	$C_0 + 0.5C_s$	$2C_0 + 0.5C_s$	$3C_0 + 0.5C_s$	
	. .	. .	. .	. .	. .	. .	. .	. . . . . .

From TABLE IV. above we can generalize the cost function

$$C(Q_c) = \begin{cases} C_s(Q_c - D) , & D = 0, 1, 2, \ldots Q_i , \\ C_0(D - Q_i) + C_sQ_R , & D = Q_i + 1, Q_i + 2, \ldots \end{cases}$$

Because $Q_c = Q_i + Q_R$ we have

$$C(Q_c) = \begin{cases} C_s(Q_i - D) + C_sQ_R, & D = 0, 1, \ldots, Q_i \\ C_0(D - Q_i) + C_sQ_R, & D = Q_i + 1, Q + 2, \ldots \end{cases} \quad (23)$$

This cost function (23) will be used for the following discussions in this chapter.

A. MINIMIZING EXPECTED COST.

When demand D is discrete and supply Q is continuous, we have the cost function as equation (23)

$$C(Q_c) = \begin{cases} C_s(Q_i - D) + C_sQ_R, & D = 0, 1, \ldots, Q_i, \\ C_0(D - Q_i) + C_sQ_R, & D = Q_i + 1, Q_i + 2, \ldots \end{cases}$$

We can see that for any supply Q_c and demand D the cost is less if Q_R is zero. Thus we can conclude that, to minimize expected cost for the case where demand D is discrete and supply Q is continuous, the optimal solution Q* must be integer-valued. Then the problem reduces to the classic discrete supply, discrete demand newsboy problem since non-integer supply values need not be considered. Optimal solutions for the aspiration level, LaPlace, and minimax cost principles of choice will also coincide with the classic problem. Again, this is for the case where demand may be satisfied <u>only</u> with discrete units.

An alternate cost structure for the continuous supply, discrete demand case would apply the unit shortage cost C_o as

to the total amount by which demand exceeds supply. Thus the cost function becomes

$$C(Q_c) = \begin{cases} C_s(Q_i + Q_R - D), & D = 0, 1, \ldots, Q_i, \\ C_0(D - Q_i - Q_R), & D = Q_i+1, Q_i+2, \ldots . \end{cases} \quad (24)$$

Now, we seek supply Q* which minimizes the expected cost. The expected cost for this problem is

$$E\{C(Q_c)\} = \sum_D \text{cost } f(D).$$

With the cost function (24) we have

$$E\{C(Q_c)\} = \sum_{D=0}^{Q_i} C_s(Q_i+Q_R-D)f(D) + \sum_{D=Q_i+1}^{\infty} C_0(D-Q_i-Q_R)f(D),$$

or

$$E\{C(Q_c)\} = \sum_{D=0}^{Q_i} C_s(Q_i-D)f(D) + \sum_{D=Q_i+1}^{\infty} C_0(D-Q_i)f(D)$$

$$+ Q_R\{\sum_{D=0}^{Q_i} C_s f(D) - \sum_{D=Q_i+1}^{\infty} C_0 f(D)\} . \quad (25)$$

The first two terms of the right-hand side of (25) are the expected cost function for the classic discrete newsboy problem. From the third term we see that $E\{C(Q_c)\}$ is linear in Q_R, and thus for an optimal solution either $Q_R = 0$ or $Q_R = 1.0$.

$$\text{If}\left[\sum_{D=0}^{Q_i} C_s f(D) - \sum_{D=Q_i+1}^{\infty} C_0 f(D)\right] > 0,\ E\{C(Q_c)\} \text{ in Equation}$$

(25) will be minimized at $Q_R = 0$, and then $Q_c = Q_i$.

$$\text{If}\left[\sum_{D=0}^{Q_i} C_s f(D) - \sum_{D=Q_i+1}^{\infty} C_0 f(D)\right] < 0\ ,\ E\{C(Q_c)\} \text{ in}$$

Equation (25) will be minimized at $Q_R = 1$, and thus $Q^*_c = Q_i + 1$. Again, we can conclude that for a minimum expected cost the optimal solution Q^* must be an integer. Thus the optimal decision rule for this problem is the same as of the standard discrete newsboy problem in equation (9)

$$F(Q^* - 1) < \frac{C_0}{C_s + C_0} < F(Q^*)\ ,\quad Q^* = 0, 1, \ldots$$

B. ASPIRATION LEVEL SOLUTION.

With the aspiration level principle of choice, it was shown earlier that

$$P\{\text{Cost} \leq A\} = P\{Q - A/C_s < D < Q + A/C_0\}\ .$$

The distribution of demand D is discrete, but for notational convenience we will interpret $F(X)$, X real, as $F(y)$, where Y is the integer part of X. Then

$$P\{\text{Cost} \leq A\} = F\{Q + A/C_0\} - F\{Q - A/C_s\}\ . \qquad (26)$$

In seeking a value of Q which maximizes this probability, we cannot differentiate $P(\text{Cost} \leq A)$ in order to find the maximum because $F(D)$ is a step function.

Since supply Q is continuous and demand D is discrete, the optimal supply Q may occur in range or interval. For the sake of the best explanation and good understanding we propose a simple numerical example.

Example 2. Suppose that the unit cost of surplus C_s is 4, the unit shortage cost C_0 is 6, and the probability distribution for demand D is

D	f(D)	F(D)
0	0.4	0.4
1	0.3	0.7
2	0.2	0.9
3	0.1	1.0

We are looking for the optimal supply Q* that maximizes $P(\text{cost} \leq A)$. Suppose that the cost aspiration level A is 5, to find the optimal solution we use the equation (26) and construct TABLE V. shown the following page. We compute $A/C_0 = .83$, and $A/C_s = 1.25$. The optimal Q* is such that $P\{Q^* \leq 5\} = F\{Q^* + .83\} - F\{Q^* - 1.25\}$ is a maximum.

TABLE V. Illustration Of How To Find The Optimal Solution From $P\{Q \leq 5\} = F\{Q + .83\} - F\{Q - 1.25\}$.

Q	F{Q +.83}	F{Q - 1.25}	Difference	
0	F{.83} = .4	F{-1.25} = 0	.4	
.17	F{1} = .7	F{-1.08} = 0	.7	
.25	F{1.08} = .7	F{-1.0} = 0	.7	
1.17	F{2} = .9	F{-0.8} = 0	.9 ←	max
1.25	F{2.08} = .9	F{0} = .4	.5	
2.17	F{3} = 1	F{0.92} = .7	.6	
2.25	F{3.08} = 1	F{1} = .7	.3	
3.	F{3.83} = 1	F{1.75} = .7	.3	

From TABLE V above, the optimal solution is

$$1.17 \leq Q^* \leq 1.25$$

Note that we choose Q so that Q + .83 and Q - 1.25 are alternately integer-valued. We can check that if Q is in the closed interval{1.17, 1.25} the cost will never exceed 5.

Maximizing cost solution and aspiration cost level solution were already discussed for this problem in case the probability of demand can be estimated. Now, we will investigate this problem in the case that the probability of demand is unknown (uncertainty). As before, principles of choice will be discussed here, namely, LaPlace and minimax cost.

C. LAPLACE SOLUTION

We already proved that the optimal expected cost rule for this problem is the same as of classic discrete case, i.e.,

$$F(Q^* - 1) < \frac{C_0}{C_s + C_0} < F(Q^*) \ .$$

Thus the decision rule for LaPlace principle of this problem must be

$$Q^* < \left(\frac{C_0}{C_s + C_0}\right)(D_{max} + 1) < Q^* + 1 \ ,$$

which is the classic discrete case (10) reviewed in Chapter II.

D. MINIMAX COST SOLUTION

Under this principle of choice we will choose Q so that the worst possible future cost will be as small as possible. In other words, we choose Q in order to minimize the maximum cost. It is clear that we will minimize the maximum cost when Q is chosen so that the cost at D = 0 is equal to the cost at $D = D_{max}$, if it is possible. In this problem, since Q is continuous, therefore we can choose Q* so that

$$C_s Q^* = C_0 (D_{max} - Q^*)$$

or

$$Q^* = \left(\frac{C_0}{D_s + C_0}\right) D_{max} \ . \qquad (27)$$

Note that this is identical with the minimax cost solution of the classic continuous case (6) found in Chapter II.

In the next chapter we will introduce and examine the "package supply" newsboy problem where demand and supply are both discrete but supply is in lots of n items.

V. OPTIMAL SOLUTIONS WHEN DEMAND AND SUPPLY ARE BOTH DISCRETE AND SUPPLY IS IN LOTS OF n ITEMS

Newsboy problems having discrete demand and supply were reviewed and displayed in Chapter II. That case, which we called the classic discrete case, assumed the same units for demand and supply. In this chapter we will discuss the case where demand and supply are both discrete, but supply occurs in lots of n demand units. This "package supply" problem is frequently encountered in application. Some examples are; the number of twelve eggs in a tray, the number of bullets in a magazine, the number of shoes in the spare box for an auto brake-drum system, and so on. For this package supply newsboy problem, we will investigate the nature of optimal solutions when the distribution of demand can be estimated (risk), and then derive formulas to determine optimal supply quantities for cases where the demand distribution cannot be estimated (uncertainty). Finally, we will compare these solutions to the classic discrete case.

We can construct the cost function as follows. Let

n = number of items in a package, where n is a constant positive integer.

S = number of packages ordered, where $S = 0, 1, 2, \ldots$ S is a decision variable.

$C(S)$ = Total cost .

Now, the supply Q is equal to nS, nS = 0, n, 2n,... . With this notation, since cost depends upon whether we have a surplus or a shortage, we can readily write the cost function for this model. The cost function is

$$C(S) = \begin{cases} C_s(nS - D), & D = 0, 1, \ldots, nS \\ C_0(D - nS), & D = nS + 1, nS + 2, \ldots \end{cases} \tag{28}$$

This cost function will be used in the derivation in this chapter.

A. MINIMIZING EXPECTED COST SOLUTION

The expected cost expression for our discrete, package supply newsboy problem is

$$E(\text{Cost}) = E\{C(S)\} = \sum_{D=0}^{nS} C_s(nS - D)f(D) + \sum_{D=nS+1}^{\infty} C_s(D - nS)f(D). \tag{29}$$

Sufficiency condition for a local minimum of a function defined for integer values of its argument are, in terms of our expected cost function with minimum at S*,

$$E\{C(S^*)\} < E\{C(S^* + 1)\}$$

and

$$E\{C(S^*)\} < E\{C(S^* - 1)\} .$$

Expanding the first inequality with the expected cost expression (29) yields:

$$\sum_{D=0}^{nS^*} C_s(nS^* - D)f(D) + \sum_{D=nS^*+1}^{\infty} C_0(D - nS^*)f(D)$$

$$< \sum_{D=0}^{n(S^*+1)} C_0\{n(S^* + 1) - D\}f(D)$$

$$+ \sum_{D=n(S^*+1)+1}^{\infty} C_0\{D - n(S^* + 1)\}f(D) .$$

After simplifying, we have

$$F\{(S^* + 1)n\} - \frac{1}{n} \sum_{D=nS^*+1}^{nS^*+n} (D - nS^*)f(D) > \frac{C_0}{C_s + C_0} . \qquad (30)$$

In the same way, expanding the second sufficiency condition with the expected cost expression (29) yields:

$$\sum_{D=0}^{nS^*} C_s(nS^* - D)f(D) + \sum_{D=nS^*+1}^{\infty} C_0(D - nS^*)f(D)$$

$$< \sum_{D=0}^{nS^*n} C_s\{n(S^* - 1) - D\}f(D) + \sum_{D=nS-n+1}^{\infty} C_0\{D - n(S - 1)\}f(D) .$$

After simplifying this yields,

$$F\{(S^* - 1)n\} + \frac{1}{n} \sum_{D=nS^*-n+1}^{nS^*} (nS^* - D) f(D) < \frac{C_0}{C_s + C_0} . \quad (31)$$

The inequalities (30) and (31) may be combined to give

$$\left[F\{(S^* - 1)n\} + \frac{1}{n} \sum_{D=nS^*-n+1}^{nS^*} (nS^* - D) f(D)\right] < \frac{C_0}{C_s + C_0}$$

$$< \left[F\{(S^* + 1)n\} - \frac{1}{n} \sum_{D=nS^*+1}^{nS^*+n} (D - nS^*) f(D)\right] . \quad (32)$$

Hence, we can find the optimal supply S^* which satisfies the double inequality (32) above.

An alternative approach for the optimal solution Q^* can be obtained as follows.

(1) Treat demand D and supply Q as in the classic discrete case.

(2) Find the optimal supply Q^* from (9) $F(Q^* - 1) < \frac{C_0}{C_s + C_0} < F(Q^*)$. If Q^* is a multiple of n, it is optimal. Otherwise, we proceed as follows.

(3) Let Q_1^* be the largest integer multiple of n which is less than Q^*, and let $S_1^* = Q_1^*/n$

(4) Let $S_2^* = S_1^* + 1$

(5) Find the expected cost of S_1^* and S_2^* from (29)

$$E\{C(S)\} = \sum_{D=0}^{nS} C_s(nS - D) + \sum_{D=nS+1}^{\infty} C_0(D - nS)f(D) .$$

(6) The optimal order quantity is the value which provides the smaller expected cost.

B. ASPIRATION LEVEL SOLUTION

For an aspiration level approach, we can obtain the solution by computing from the cost matrix in a manner identical to that done for the classic discrete case in Chapter II. Thus, we have the cost array shown on TABLE VI.

TABLE VI. The Cost Array For The Case Where Demand And Supply Are Both Discrete And Supply Is In Lots Of n Items

	f(D)	f(0)	f(1)	f(2)	f(3)		f(n)	f(n+1)	f(n+2)
S	Q	0	1	2	3	.	n	n+1	n+2
0	0	0	C_0	$2C_0$	$3C_0$	.	nC_0	$(n+1)C_0$	.
1	n	$C_s n$	$C_s(n-1)$	$C_s(n-2)$	.	.	0	C_0	$2C_0$
2	2n	$C_s \cdot 2n$	$C_s(2n-1)$	.	.	.	.	.	.
3	3n	.	.	.	.	.	.	.	.
4	4n	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.

When the problem is laid out in the matrix format, it becomes a straightforward matter to determine, for each value of S, the probability that cost will not exceed the aspired value A. The optimal value S* would be the one that provides the maximum of these probabilities.

As an alternate approch, we can obtain a more general aspiration level solution procedure by the following derivation. From FIGURE 3 we see that

$$P\{\text{Cost} \leq A\} = P\{nS - A/C_s \leq D \leq nS + A/C_0\} \ ,$$
$$D = 0,\ 1,\ 2, \ldots \ . \qquad (33)$$

We can simplify equation (33) as

$$P\{\text{Cost} \leq A\} = F\{nS + A/C_0\} - F\{nS - A/C_s\} \ .$$

Since demand D is discrete with D = 0, 2, 3,..., $nS + A/C_0$ must be rounded down to an integer I_2, and $nS - A/C_s$ must be rounded up to integer $I_1 \geq 0$, so

$$P\{\text{Cost} \leq A\} = F\{I_2\} - F\{I_1\}$$

or

$$P\{\text{Cost} \leq A\} = \sum_{D=I_1}^{D=I_2} f(D) \ . \qquad (34)$$

The optimal solution S* is such that the equation (34)

$$P\{\text{Cost} \leq A\} = \sum_{D=I_1}^{D=I_2} f(D) \text{ is a maximum.}$$

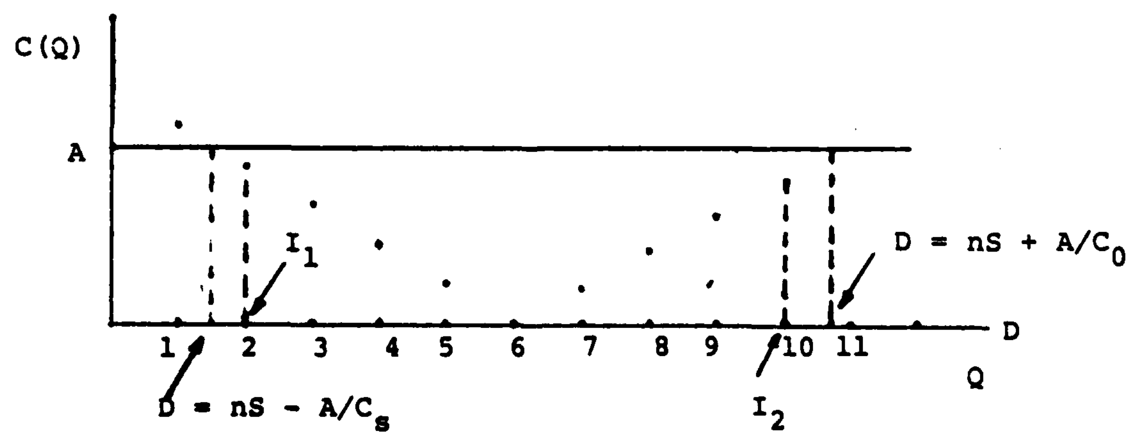

FIGURE 3. $P\{\text{Cost} \leq A\} = P\{nS - A/C_s \leq D \leq nS + A/C_0\}$, given n = 3.

To find the optimal solution S*, we can construct a table and as TABLE VII.

TABLE VII. Illustration of the Method to find an Optimal Aspiration Level Solution S* for the Package Supply Problem.

S	Q=nS	$nS-A/C_s$	$nS+A/C_0$	I_1	I_2	$\sum_{D=I_1}^{D=I_1}$	f(D)
0	.	.	.	.	.	.2	
1	.	.	.	.	.	.4	
S*→ 2	.	.	.	.	.	.8 ←	max.
3	.	.	.	.	.	.3	

In the following sections we will discuss the decision under uncertainty for this model. First, we will derive the LaPlace solution, and then the minimax cost solution.

C. LAPLACE SOLUTION

The LaPlace approach, as described in Chapter II, is to assume that demand is uniformly distributed over the demand interval, and then use an expected value solution. Thus, for this model, we would assume that the distribution of demand is uniform (discrete). Thus with an estimated of the maximum demand D_{max}

$$f(D) = \frac{1}{D_{max} + I} , 0, 1, \ldots, D_{max} ,$$

and the distribution function is

$$F(D) = \frac{D+1}{D_{max} + 1} .$$

The optimal expected value rule (32) developed at the beginning of this chapter is

$$F\{(S^*-1)n\} + \frac{1}{n} \sum_{D=nS^*-n+1}^{nS^*} (nS^*-D)f(D) < \frac{C_0}{C_s + C_0} < F\{S^*+1)n\}$$

$$- \frac{1}{n} \sum_{D=nS^*+1}^{nS^*+n} (D-nS^*)f(D) .$$

Substituting F(D) and F(D) leads to

$$\frac{(S^*-1)n+1}{D_{max} + 1} + \frac{1}{n} \sum_{D=nS^*-n+1}^{nS^*} (nS^*-D)\frac{1}{D_{max} + 1} < \frac{C_0}{C_s + C_0} < \frac{(S^*+1)n+1}{D_{max} + 1}$$

$$- \frac{1}{n} \sum_{D+nS^*+1}^{nS^*+n} (D-nS^*) \frac{1}{D_{max} + 1} .$$

or

$$(S^*-1)n+1 + \frac{1}{n} \sum_{D=nS^*-n+1}^{nS^*} (nS^*-D) < \frac{C_0}{C_s + C_0} (D_{max}+1) < (S^*+1)n+1$$

$$- \frac{1}{n} \sum_{D=nS^*+1}^{nS^*+n} (D-nS^*) \ . \tag{35}$$

To simplify this equation (8) we use the notion that

$$\sum_{D=a}^{b} D = \frac{b^2-a^2+a+b}{2} \text{ and } \sum_{a}^{b} C = (b-a+1)C$$

where D = 0, 1, 2,..., and a, b, C, are constant non-negative integers. The left hand side of (35) becomes

$$(S^*-1)n + 1 + \frac{1}{n} n^2S^* - \frac{1}{2n}\{(nS^*)^2 - (nS^*-n+1)^2$$
$$+nS^* - n + 1 + nS^*\}$$

which is simplified so that the left inequality is

$$nS^* - \frac{1}{2} (n-1) < \frac{C_0}{C_s + C_0} \cdot (D_{max} + 1) \tag{36}$$

The right hand side of (35) becomes

$$(S^*+1)n + 1 + \frac{1}{n} n^2S^* - \frac{1}{2n}\{(nS^*+n)^2 - (nS^*1)^2 + nS^* + 1 + nS^*+ n\}.$$

We can simplify this by the same manner as before and we have

$$nS^* + \frac{1}{2} (\ddot{n}+1) > \frac{C_0}{C_s+ C_0} (D_{max} + 1) . \tag{37}$$

The inequalities (36) and (37) lead to

$$nS^* - \frac{1}{2} (n-1) < \frac{C_0}{C_s+ C_0} (D_{max} + 1) < nS^* + \frac{1}{2} (n+1) \tag{38}$$

as the optimal decision rule of LaPlace solution for this "package supply" problem. Note that if the value of n is unity, the inequality (38) becomes

$$S^* < \frac{C_0}{C_s+ C_0} (D_{max} + 1) < S^* + 1 ,$$

which is identical to the optimal decision rule (11) for the classic discrete case in Chapter II.

As an alternative approach, we can obtain the optimal solution for the LaPlace principle of choice as follows:

(1) Treat supply Q with units of 0, 1, 2,... , so that the model becomes the classic discrete one.

(2) Use optimal decision rule (11) in Chapter II for the LaPlace solution

$$Q^* < \frac{C_0}{C_s + C_0}(D_{max} + 1) < Q^* + 1 .$$

If Q^* is the multiple of n, then Q^* is optimal solution; if not we proceed as follows.

(3) Let Q_1^* be the largest integer multiple of n which less than Q^*, and let $S_1^* = Q_1^* / n$

(4) Let $S^*_2 = S_1^* + 1$.

Now, the optimal solution is either S_1^* or S_2^*, which may be determined by comparing the expected cost of S_1^* and S_2^* from (29)

$$E\{C(S)\} = \sum_{D=0}^{nS} C_s(nS-D)f(D) + \sum_{D=nS+1}^{\infty} C_0(D-nS)f(D) .$$

The one which provides the smaller expected cost is the optimal solution. Next we will investigate and derive a decision rule for the minimax cost solution.

D. MINIMAX COST SOLUTION

This approach for decisions under uncertainty, as mentioned in Chapter II, is to minimize the maximum cost. We assume that the only thing we know about demand is D_{max}.

For this model, the newsboy cost function is

$$C(S) = \begin{cases} C_s(nS - D), & D = 0, 1, \ldots, nS \\ C_0(D - nS), & D = nS + 1, nS + 2, \ldots D_{max} \end{cases}$$

In order for the maximum costs to be minimized at S*, the following inequalities must hold:

$$\text{Maximum } C(S^*) < \text{Maximum } C(S^* + 1) \qquad (39)$$

and

$$\text{Maximum } C(S^*) < \text{Maximum } C(S^* - 1) \qquad (40)$$

The cost function for package supply S will have its maximum value at either demand equal to zero or demand equal to D_{max}. Thus the maximum cost will be either $C_s nS$ or $C_0(D_{max} - nS)$, and the first condition (39) may be written as

$$\text{Max}\{C_s nS^*, C_0(D_{max} - nS^*)\} < \text{Max}\{C_s n(S^*+1), C_0(D_{max} - n(S^*+1))\}. \qquad (41)$$

Now

$$C_s nS^* < C_s n(S^* + 1)$$

and

$$C_0\{D_{max} - n(S^* + 1)\} < C_0\{D_{max} - nS^*\}.$$

Applying these two inequalities to the inequality (41), the first condition becomes

$$C_0\{D_{max} - nS^*\} < C_s\{n(S^* + 1)\}.$$

Adding $C_0\{n(S^* + 1)\}$ to both sides, we have

$$C_0\{D_{max} - nS^* + n(S^* + 1)\} < n(S^* + 1)(C_s + C_0) ,$$

or

$$n(S^* + 1) < \frac{C_0}{C_s + C_0}(D_{max} + n) \ . \tag{42}$$

as a useful form of the first condition for a minimax cost solution at S*.

The second condition (40) may also be written as

$$\text{Max}\{C_s nS^*, C_0(D_{max} - nS^*)\} < \text{Max}\{C_s n(S^*-1), C_0(D_{max} - n(S^*-1))\}. \tag{43}$$

Now

$$C_s nS^* > C_s n(S^* - 1) \ ,$$

and

$$C_0\{D_{max} - n(S^* - 1)\} > C_0\{D_{max} - nS^*\} \ .$$

Applying these two inequalities to the inequality (43), the second condition becomes

$$C_s nS^* < C_0\{D_{max} - n(S^* - 1)\} \ .$$

Adding $C_0 nS^*$ to both sides, we have

$$nS^*(C_s + C_0) < C_0\{D_{max} - n(S^* - 1) + nS^*\},$$

or

$$nS^* < \frac{C_0}{C_s + C_0}\{D_{max} + n\} \tag{44}$$

as a useful form of the second condition for a minimax cost solution at S*. Finally the inequality (42) and the inequality (44) lead to

$$nS^* < \frac{C_0\{D_{max} + n\}}{C_s + C_0} < n(S^* + 1) \quad . \tag{45}$$

as the optimal decision rule for the minimax cost solution.

The package supply newsboy problem investigated in this chapter is similar to the classic discrete problem, and the derivations and optimal solutions of both problems are comparable. In the next chapter we will examine a continuous problem where supply Q is in lots of size n, where n can be any positive real number.

VI. OPTIMAL SOLUTIONS WHEN DEMAND AND SUPPLY ARE BOTH CONTINUOUS BUT SUPPLY IS IN LOTS OF SIZE n

In previous chapters we saw several different variations of the newsboy problem. In this chapter we will introduce another interesting one. It is the case where demand and supply are both continuous but supply is in lots of size n. This problem is similar to the case where demand is continuous and supply is discrete investigated in Chapter III. The difference is that the supply is in lots of size n, where n can be any positive real number. (If the value of n is unity, this case is identical to the case found in Chapter III.) This "package supply" newsboy problem is frequently encountered in application. An example is that of how many cans (each contains 3.5 gallons) of lubricating oil should be held in stock if the demand for oil is a continuous unknown variable.

We can construct the cost function of this "package supply" problem as follows. Let

n = package size, is a constant positive real number

S = number of packages ordered, a decision variable limited to non-negative integers.

Here the supply Q is equal to nS, where $nS = 0, n, 2n, \ldots$ With this notation and the previous assumption of shortage cost and surplus cost, we can readily write the cost functions for this model. The cost function is

$$\text{cost} = C(S) = \begin{cases} C_s(nS - D), & 0 \leq D \leq nS, \\ C_0(D - nS), & nS \leq D, \end{cases} \qquad (46)$$

where C_s is cost per unit of supply that exceeds demand and C_0 is cost per unit of demand that exceeds supply.

As mentioned earlier, this "package supply" problem is similar to that one discussed in Chapter III. The easiest way to solve this problem is to change the unit of demand D to the unit of supply Q (package or can). The transformation as follows:

Let $D' = \frac{D}{n}$. Thus D' is in units of package size (or can). Now demand D' is continuous and supply S is discrete S = 0, 1, 2,..., which is identical to the model examined in Chapter III. We must also change the probability density function of demand f(D), and correct the cost function. The inverse of transformation is

$$D = nD' ,$$

and the Jacobian is

$$\frac{dD}{dD'} = n .$$

By transformation principle we have

$$f(D') = nf(D)$$

as the probability density function of D'. The unit surplus cost becomes nC_s and the unit shortage cost becomes nC_0.

From this we can rewrite the cost function as

$$\text{Cost} = C(S) = \begin{cases} nC_s(S - D') \ , & 0 < D' < S \ , \\ nC_0(D' - S) \ , & S < D', \quad S = 0, 1, \ldots \ . \end{cases} \tag{47}$$

we can solve this "package supply" easily by applying the appropriate optimal decision rules derived in Chapter III for the case where demand is continuous and supply is discrete, as a numerical example.

Example 3. Suppose the probability density function of demand is $f(D) = 4e^{-4D}$ $D \geq 0$, the unit surplus cost is 5 and the unit shortage cost is 7.5, and the ordering unit is 2.5 demand units. Thus $n = 2.5$.

Here we have $D' = \frac{D}{2.5}$ and $\frac{dD}{dD'} = 2.5$ as the Jacobian of transformation. The density function $f(D') = nf(D)$ becomes $f(D') = 2.5(4e^{-4(2.5D')})$ or $e^{-D'}$ where $D = 2.5D'$, and the unit surplus cost and unit shortage cost become 2 and 3 respectively.

We have shown ways of obtaining optimal solutions to the fourth and last of the newsboy problems we set out to investgate. In the next chapter we will give some conclusions to this paper and some suggestions for further work on variations of the newsboy problem.

VII. CONCLUSIONS AND SUGGESTIONS FOR FURTHER WORK.

From this thesis, we achieved decision rules for two "mixed newsboy" problems and two "package supply" newsboy problems. The principles of choice considered were expected cost and aspiration level for decisions under risk, and LaPlace and minimax cost for decisions under uncertainty. We began by reviewing the notion of decision making, showing the optimal decision rules for the standard newsboy problem under each of the four principles of choice. Then we considered two mixed newsboy problems, one with continuous supply and discrete demand, and the other with discrete supply and continuous demand. Thereafter, we discussed and investigated two "package supply" newsboy problems. It was found that sufficiency conditions for the local minimum of a function (finite difference equations[2]) played an important role in investigating expected cost solutions for these problems. We could not apply ordinary differential calculus because both demand and supply were not continuous.

Some of the decision rules we achieved in this paper are not quite practical because of the intractability of the formulas, however we, in some cases, were able to show alternate ways to reach the optimal solution by applying the standard decision rules for the classic problem.

We found that the problem where demand and supply are both continuous but supply is in lots of size n is similar

to the discrete supply, continuous demand problem, and thus we showed an easy way to reach the optimal solution by simply transforming the problem.

In the investigations for achieving decision rules for "mixed" and "package supply" newsboy problems in this paper, only linear cost functions are used. In many applications, consideration of the kinds of costs involved suggests that a U-shaped cost curve is required[4]. With this consideration in view, the cost functions may sometimes be approximated with reasonable accuracy by a positive definite quadratic form. For further work, it is recommended that quadratic cost functions be employed in the four newsboy problems in this paper, and optimal solutions sought.

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